15

LÉVY PROCESS, LÉVY FLIGHTS, AND WEIERSTRASS RANDOM WALK

Considering random processes one can model different situations with different levels of the dependence of events at different time instants, with different levels of memory and correlation between the events, and so on. This kind of an axiomatic approach to the kinetics of dynamical systems is hardly acceptable unless the intrinsic dynamical features will be involved into the random process model in an explicit way. From the very beginning of the investigation of chaos, there were observations of an intermittent character of the time behaviour of chaotic trajectories. It became clear that the models of Gaussian type process and normal diffusion were not always valid and there were many typical situations when simplified models of independent or weakly dependent random events must be abandoned (*Note 15.1*).

Different careful observations of Hamiltonian systems with chaotic dynamics impose the necessity of extending the tools to explain how the dynamics should be described and what kind of approximations fit better. The Gaussian distribution links to the so-called *large number law* are well-known: the sum of independent random variables is distributed in the same way as any of them. The uniqueness of the Gaussian distribution was reconsidered by Lévy (1937) who had formulated a new approach that can also be applied to distributions with infinite second moment. There is a nice description of the history of Lévy's discovery, as well as its relation to other probabilistic theories and to the St Petersburg paradox of Daniel Bernoulli in Montroll and Shlesinger (1984). It happened that the Lévy distribution and Lévy processes had a strong impact on different areas of scientific analysis including not only the probability theory, but also physics, economics, financial mathematics, geophysics, dynamical systems, and so on. Mandelbrot (1982) has indicated numerous applications of the Lévy distributions and coined a notion of *Lévy flights*.

It became clear that the ideas related to the Lévy processes can be important in the analysis of chaotic dynamics after some modification is applied. In this chapter, brief information about the Lévy processes will be introduced with an emphasis on which features of the process can be or cannot be used for the dynamically chaotic trajectories (*Note 15.2*).

15.1 Lévy distribution

Let $P(x)$ be a normalized distribution of a random variable x, i.e.

$$\int_{-\infty}^{\infty} P(x)dx = 1 \tag{15.1}$$

with a characteristic function

$$P(q) = \int_{-\infty}^{\infty} dx\ e^{iqx} P(x). \tag{15.2}$$

Consider two different random variables x_1 and x_2 and their linear combination

$$c\,x_3 = c_1 x_1 + c_2 x_2, \qquad c, c_1, c_2 > 0. \tag{15.3}$$

The law is called stable if all x_1, x_2, x_3 are distributed due to the same function $P(x_j)$. Gaussian distribution

$$P_G(x) = (2\pi\sigma_d)^{-1/2} \exp\left(-\frac{x^2}{2\sigma_d}\right) \tag{15.4}$$

is an example of the stable distribution with a finite second moment

$$\sigma_d = \langle x^2 \rangle \tag{15.5}$$

(compare to (14.39)). Another class of solutions was found by Lévy (1937).

Let us write the equation

$$P(x_3)dx_3 = P(x_1)P(x_2)\delta\left(x_3 - \frac{c_1}{c}x_1 - \frac{c_2}{c}x_2\right) dx_1 dx_2, \tag{15.6}$$

where the condition (15.3) has been used. Following the definition (15.2), one can write the equation for characteristic function

$$P(cq) = P(c_1 q)P(c_2 q) \tag{15.7}$$

or

$$\ln P(cq) = \ln P(c_1 q) + \ln P(c_2 q). \tag{15.8}$$

Equations (15.7) and (15.8) are functional ones with an evident solution

$$\ln P_\alpha(cq) = (cq)^\alpha = c^\alpha e^{-i\frac{\pi}{2}\alpha(1-\operatorname{sign} q)} |q|^\alpha \tag{15.9}$$

and condition

$$\left(\frac{c_1}{c}\right)^\alpha + \left(\frac{c_2}{c}\right)^\alpha = 1 \tag{15.10}$$

which consists of an arbitrary parameter α.

The distribution $P_\alpha(x)$ with the characteristic function

$$P_\alpha(q) = \exp(-c|q|^\alpha) \tag{15.11}$$

is known as *Lévy distribution* with the *Lévy index* α. For $\alpha = 2$ we arrive at the Gaussian distribution $P_G(x)$. An important condition introduced by Lévy is

$$0 < \alpha \leq 2 \tag{15.12}$$

which guarantees positiveness of

$$P_\alpha(x) = \int dq\, e^{iqx} P_\alpha(q). \tag{15.13}$$

The case $\alpha = 1$ is known as Cauchy distribution

$$P_1(x) = \frac{c}{\pi}\,\frac{1}{x^2 + c^2}. \tag{15.14}$$

An important case is the asymptotic of large $|x|$

$$P_\alpha(x) \sim \frac{1}{\pi}\alpha\, c\,\Gamma(\alpha)\sin\frac{\pi\alpha}{2}\,\frac{1}{|x|^{\alpha+1}}, \qquad (0 < \alpha < 2) \tag{15.15}$$

(*Note 15.3*). It shows that the moments of $P_\alpha(x)$

$$\langle x^m \rangle = \int_{-\infty}^{\infty} dx x^m P_\alpha(x) \tag{15.16}$$

diverge for

$$m \geq \alpha, \tag{15.17}$$

that is, $\langle x^2 \rangle = \infty$ and that is why the large number law does not work for the considered situation.

15.2 Lévy process

There are many different ways to introduce the Lévy process, i.e. a time-dependent process that at an infinitesimal time has the Lévy distribution of the process variable (Lévy (1937); Gnedenko and Kolmogorov (1959); Feller (1957); Uchaikin and Zolotarev (1999); Montroll and Shlesinger (1984)). Here we use a simplified version for the infinitely divisible processes described below.

Consider the transition probability density $P(x_0, t_0; x_N, t_N)$ that satisfies the chain equation

$$P(x_0, t_0; x_N, t_N) = \int dx_1 \ldots dx_{N-1} P(x_0, t_0; x_1, t_1) \times P(x_1, t_1; x_2, t_2 \ldots P(x_{N-1}, t_{N-1}; x_N, t_N) \tag{15.18}$$

and put

$$t_{j+1} - t_j = \Delta t, \quad (\forall j); \qquad t_N - t_0 = N\Delta t. \tag{15.19}$$

Assume that the process is uniform in time and space, i.e.

$$P(x_j, t_j; x_{j+1}, t_{j+1}) = P(x_{j+1} - x_j; t_{j+1} - t_j) = P(x_{j+1} - x_j; \Delta t). \tag{15.20}$$

Then (15.18) transforms into

$$P(x_N - x_0; N\Delta t) = \int dy_1 \dots dy_N P(y_1, \Delta t) \dots P(y_N; \Delta t), \tag{15.21}$$

where $y_j = x_j - x_{j-1}$; $j \geq 1$.

By introducing the characteristic functions

$$\begin{aligned} P(q) &= \int dy_j \; e^{iqy_j} P(y_j; \Delta t), \qquad (j \neq N), \quad (\forall j), \\ P_N(q) &= \int dy^{(N)} \; e^{iqy^{(N)}} P(y^{(N)}; N\Delta t), \quad y^{(N)} = \sum_1^N y_j = y_N - y_0, \end{aligned} \tag{15.22}$$

we obtain from (15.21)

$$P_N(q) = [P(q)]^N \tag{15.23}$$

(see Problem 15.1). Following the concept of stable distributions and using the expression (15.7), let us consider $P(q)$ as a function of two parameters α and c that will be defined later. Namely, change the notation

$$P(q) \to P_\alpha(q; \Delta c); \quad P_N(q) \to P_\alpha(q; c_N), \tag{15.24}$$

where Δc or c_N should replace c in (15.11). These equations are consistent if

$$c_N = N\Delta c = N\Delta t \cdot \frac{\Delta c}{\Delta t} \equiv cN\Delta t = ct \tag{15.25}$$

and (15.23) with notations (15.24) takes the form

$$P_\alpha(q; ct) = \exp(-cN\Delta t|q|^\alpha) \tag{15.26}$$

with $t = N\Delta t$ and $t_0 = 0$. In the limit $\Delta t \to 0$, $N \to \infty$, $N\Delta t = t$, and $c = \Delta c/\Delta t$ we arrive to the *characteristic function* of the *Lévy process*:

$$P_\alpha(q, t) = \exp(-ct|q|^\alpha). \tag{15.27}$$

The original Lévy process can be written as the inverse Fourier transform of (15.27)

$$P_\alpha(x,t) = \int dq\ e^{iqx - ct|q|^\alpha} \tag{15.28}$$

with the asymptotics for $|x| \to \infty$ similar to (15.15)

$$P_\alpha(x,t) \sim \frac{1}{\pi}\alpha c\Gamma(\alpha)\sin\frac{\pi\alpha}{2}\cdot\frac{t}{|x|^{\alpha+1}}. \tag{15.29}$$

From (15.28) we have for the moment of order m (not necessarily integer)

$$\langle |x|^m \rangle = \infty, \quad m \geq \alpha \tag{15.30}$$

and, since $\alpha < 2$,

$$\langle x^2 \rangle = \infty \tag{15.31}$$

for any t, in contrary to the Gaussian process.

There are different generalizations of Lévy distributions and Lévy processes that can be useful in applications. Mainly, they are related to the anisotropy of the distributions. For example

$$P_\alpha(q,\xi,c) = \exp\left\{-c|q|^\alpha[1 + i\xi\,\mathrm{sign}\,q\cdot\tan\left(\frac{\pi\alpha}{2}\right)\right\}, \qquad \alpha \neq 1; \quad 0 < \alpha < 2 \tag{15.32}$$

with

$$\xi = \frac{c^+ - c^-}{c^+ + c^-}, \quad c = a_\alpha(c^+ + c^-), \tag{15.33}$$

where a_α is a constant defined by the normalization condition and for $\alpha = 1$, $\tan(\pi\alpha/2)$ should be replaced by $(\pi/2)\ln|q|$ (*Note 15.4*). Distribution (15.32) provides the asymptotics

$$P_\alpha(x,\xi,c) \sim \frac{c^\pm}{|x|^{\alpha+1}}. \tag{15.34}$$

Streaming distribution

$$P_\alpha(x,t) \sim \frac{\mathrm{const}}{|x - vt|^{\alpha+1}} \tag{15.35}$$

was considered in Montroll and Shlesinger (1984). Another important generalization, the so-called Lévy walks (Shlesinger *et al.* (1987)) will be considered in Chapter 18.

15.3 Poincaré recurrences and Feller's theorems

In this section we describe some results formulated in Feller (1949). They show a connection between the Lévy processes and Poincaré recurrences. Although these results are not related directly to dynamical systems, the Poincaré recurrences distribution plays an important role in kinetics as we will see in following chapters. From that point, Feller's theorems are 'half-way' to the dynamics.

Consider a small domain A and recurrences to A. The recurrence time is the time interval between two subsequent crossings of the boundary of A by a trajectory of the particle on its way out of A. In bounded Hamiltonian dynamics the sequence of recurrence times $\{t_j\} \equiv t_1, t_2, \ldots, t_n, \ldots$ is infinite for non-periodic orbits. It is assumed that t_j are mutually independent and they belong to the same class of events with all identical probability distribution function

$$P_{\text{rec}}(\tau) = \text{Prob}\{t_k = \tau\}, \quad (\forall k), \tag{15.36}$$

i.e. independent on k. The integrated probability of recurrences is

$$P_{\text{rec}}^{\text{int}}(t) = \int_0^t d\tau P_{\text{rec}}(\tau) \tag{15.37}$$

and

$$\Phi_S(t) = 1 - P_{\text{rec}}^{\text{int}}(t) = \int_t^\infty d\tau P_{\text{rec}}(\tau) \tag{15.38}$$

has a meaning of a probability that the recurrence time is $\geq t$, that is, the survival probability (probability to survive time t in A).

Feller (1949) introduced two additional characteristics of the recurrences chain: sum of n recurrence times

$$S_n = t_1 + \cdots + t_n \tag{15.39}$$

and number of the recurrences N_t during time interval $(0, t)$. An evident connection between them is

$$\text{Prob}\{N_t \geq n\} = \text{Prob}\{S_n \leq t\}. \tag{15.40}$$

Due to the Kac lemma the mean recurrence time τ_{rec} is finite. Then under the conditions of independence of t_j and finiteness of τ_{rec}, two following theorems are valid (Feller (1949)):

1. If $\sigma_0^2 < \infty$ then for every fixed ξ

$$\text{Prob}\{S_n - n\tau_{\text{rec}} \leq n^{1/2}\sigma_0\xi\} \to \Phi(\xi) = (2\pi)^{-1/2} \int_{-\infty}^{\xi} dy \exp\left(-\frac{y^2}{2}\right),$$

$$\text{Prob}\left\{N_t \geq \frac{t}{\tau_{\text{rec}}} - t^{1/2}\frac{\sigma_0\xi}{\tau_{\text{rec}}^{3/2}}\right\} \to \Phi(\xi), \tag{15.41}$$

where

$$\sigma_0^2 = \langle (t_k - \langle t_k \rangle)^2 \rangle = \langle (t_k - \tau_{\rm rec})^2 \rangle \tag{15.42}$$

and the connection from (15.40) has been used

$$n\tau_{\rm rec} + n^{1/2}\sigma_0 \xi = t. \tag{15.43}$$

A simple meaning of this theorem is that fluctuations of the recurrence time from its mean value $\tau_{\rm rec}$ are distributed due to the Gaussian law.

2. Let $\Phi_S(t)$ in (15.38) has the asymptotics

$$\Phi_S(t) = \frac{1}{t^\alpha} h(t), \quad 0 < \alpha < 2 \tag{15.44}$$

with

$$\lim_{x \to \infty} \frac{h(cx)}{h(x)} = 1.$$

Then for $1 < \alpha < 2$

$$\text{Prob}\left\{ N_t \geq \frac{t}{\tau_{\rm rec}} - \frac{b_t}{\tau_{\rm rec}^{(1+\alpha)/\alpha}} \xi \right\} \to P_\alpha(\xi), \tag{15.45}$$

that is, to the Lévy distribution with $P_\alpha(\xi)$ from (15.15) and b_t to be obtained from the equation

$$\Phi_S(b_t) \sim \frac{1}{t^{1/\alpha}}. \tag{15.46}$$

Distribution (15.45) is *the only possible non-normal distribution for* N_t and $1 < \alpha < 2$.

The last statement of the Feller's theorem does not leave us any possibility to escape the Lévy distribution for physical problems since the conditions of the theorem are fairly broad. We do not put the case $0 < \alpha < 1$ since it is forbidden due to the Kac lemma. Indeed, comparing (11.19), (15.38), and (15.44) we obtain $\alpha = \gamma - 1$ and the condition $\gamma > 2$ means $\alpha > 1$.

The presented theorems extend our information about a universality of distribution of the recurrence time fluctuations as random variables: for a finite dispersion σ_0^2 they are distributed due to the Gaussian law, and for an infinite dispersion due to the Lévy law. Although the Gaussian or Lévy distributions can sometimes be a good approximation, the chaotic Hamiltonian dynamics show much more complicated processes.

15.4 Lévy flights and conflict with dynamics

The analysis of Chapters 11–13 shows that the distribution of Poincaré recurrences provides fairly detailed information about trajectories in phase space. In the previous section, a fairly strong statement imposes an alternative kind of

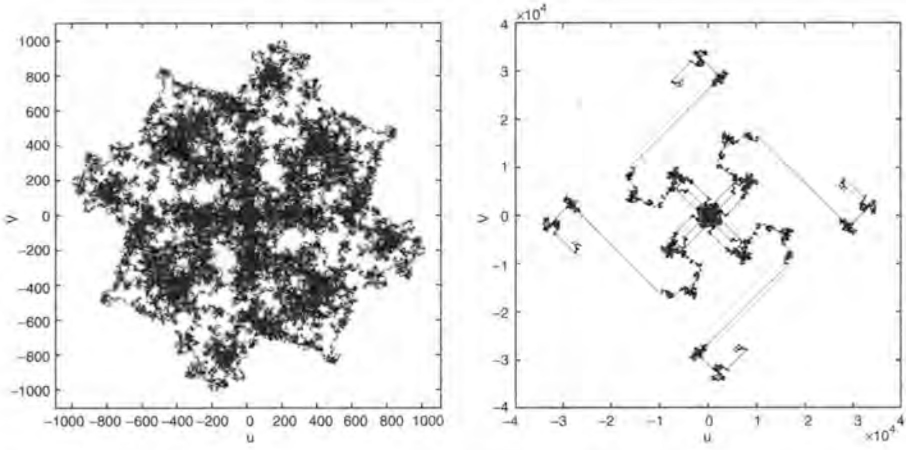

FIG. 15.1. Two samples of trajectories for the web map: trajectory without flights (left) for $K = 5.001$; and trajectory with flights (right) for $K = 6.349972$.

time behaviour of independent events linked them to either Gaussian or Lévy type processes. What is the actual type of the process that is related to real dynamical systems?

Let us first provide a few demonstrations of trajectories of different types.

Figure 15.1 (Zaslavsky and Niyaziv (1997)) shows two different kinds of trajectories for the web map (5.35). They correspond to slightly different values of the control parameter K.

The qualitative difference is more drastic. On the left figure the trajectory makes a more or less uniform random walk in phase space, while on the right figure the trajectory exhibits 'flights' of a length of 10^3 or more. Similar quasi-regular pieces exist for the standard map, billiards, etc. These deviations from the uniform distribution of trajectories in phase space lead to a non-Gaussian diffusion with power-like asymptotical distribution of the displacements of trajectories.

Intermittency can be of different kind and, depending on that, different flights can appear. The example in Fig. 15.2 shows parabolic shape pieces of trajectories in phase space. They appear near a constant value of acceleration, that is, near the accelerator mode trajectory.

Another example is related to the '*Cassini billiard*' which is a square table with the Cassini oval shape of the scatterer inside the square:

$$(x^2 + y^2)^2 - 2c^2(x^2 - y^2) - (a^4 - c^4) = 0,$$

where a, c are parameters of the oval.

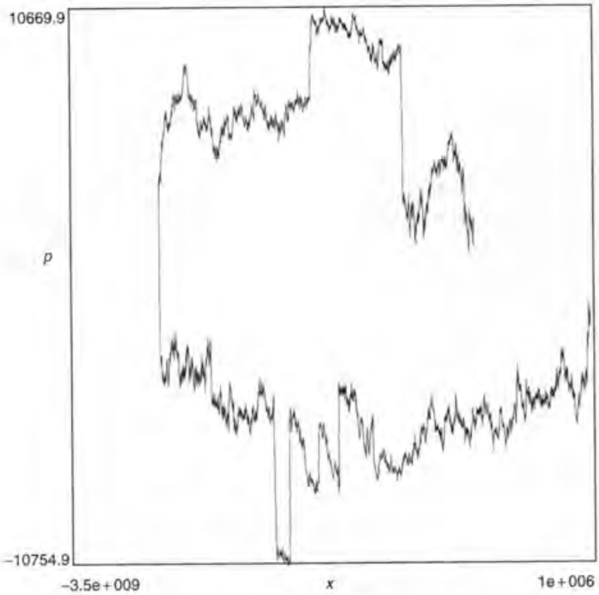

FIG. 15.2. Parabolic flights for the standard map $K = 6.908745$.

In Fig. 15.3 we can see stickiness in phase space (dark strips in part (a)) and flights in part (b) (*Note 15.5*). Magnifications of the phase space area near a sticky island are shown in Fig. 15.4. A sequence of zooms display a sequence of islands-around-islands $4 - 8 - 4 - 8 - \cdots$. Two parameters can characterize the hierarchical dynamical trap related to the sticky area near the island's boundary: ΔS_k-area of an island of k-th generation, T_k-period of the last invariant curve in an island of k-th generation. All these parameters are presented in Table 15.1. This also includes the initial two islands (in Fig. 15.3 we show only one of two central islands).

Table 15.1 also displays the values of proliferation coefficient q_k for the k-th generation, and the area

$$\delta S_k = q_k \Delta S_k \tag{15.47}$$

of all islands of k-th generation. In accordance with Section 12.2, two scaling parameters can be introduced in order to describe a self-similarity of the islands' hierarchy:

$$\begin{aligned} \lambda_S^{(k)} &= \frac{\delta S_k}{\delta S_{k-1}}, \\ \lambda_T^{(k)} &= \frac{T_k}{T_{k-1}} \end{aligned} \tag{15.48}$$

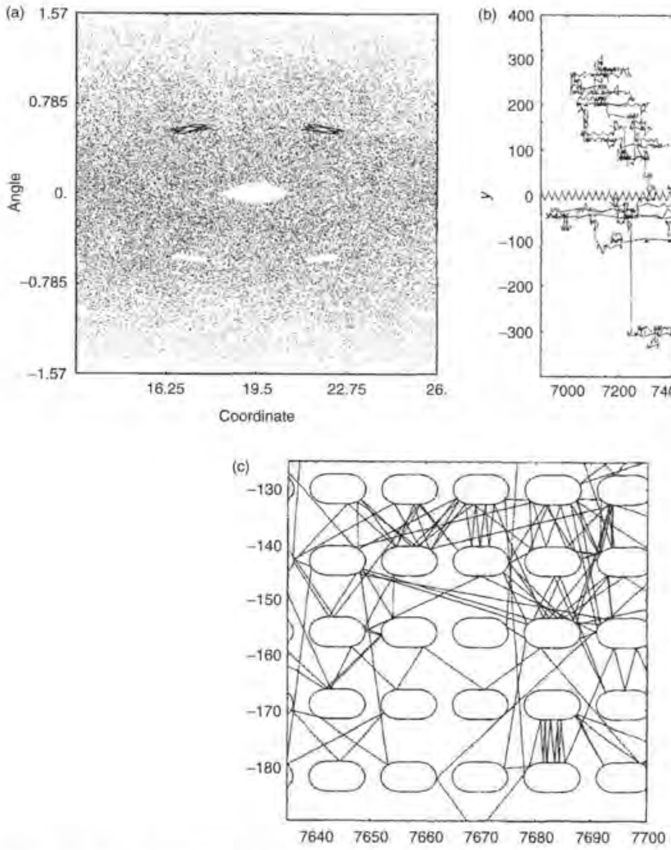

FIG. 15.3. One trajectory of the Cassini billiard: (a) Poincaré section in the phase space $(x, \cos^{-1} v_x)$ on the torus; (b) the same trajectory in the coordinate space (x, y) of the corresponding Lorentz-type gas, i.e. a periodically continued set of scatterers in (x, y) directions; (c) magnification of the same as in (b) trajectory in (x, y) space. Parameters are $a = 4.030952$; $c = 3$.

and the constant value $q_k = q$ $(\forall k \geq 1)$ means the existence of constant (approximately) values of scaling parameters

$$\lambda_S^{(k)} = \lambda_S, \quad \lambda_T^{(k)} = \lambda_T, \qquad (\forall k \geq 1). \tag{15.49}$$

In the case in Figs. 15.3 and 15.4 for the Cassini billiard we are faced with a new situation. It has two values of $\lambda_S^{(1,2)}$ and $\lambda_T^{(1,2)}$ which are found in Table 15.1:

$$\begin{aligned} \lambda_T^{(1)} \sim 7.4, \quad \lambda_T^{(2)} \sim 4.2, \\ \lambda_S^{(1)} \sim 0.017, \quad \lambda_S^{(2)} \sim 0.21, \end{aligned} \tag{15.50}$$

TABLE 15.1 Parameters ΔS_k, T_k of the island's hierarchy in the sequence $4-8-4-8-\cdots$.

k	q_k	T_k	T_k/T_{k-1}	ΔS_k	$\Delta S_k/\Delta S_{k-1}$	δS_k	$\delta S_k/\delta S_{k-1}$
0	2	16.36	–	1.47×10^{-2}	–	2.94×10^{-2}	
1	4	118	7.21	3.96×10^{-3}	2.69×10^{-2}	3.17×10^{-2}	1.08
2	8	508.9	4.31	8.53×10^{-6}	2.15×10^{-3}	5.46×10^{-4}	0.017
3	4	3910	7.69	4.4×10^{-7}	5.2×10^{-2}	1.1×10^{-4}	0.21
4	8	15 740	4.02	0.96×10^{-10}	2.2×10^{-3}	2.0×10^{-6}	0.018

where the mean values are taken and $\delta S_1/\delta S_0$ is skipped since it does not correspond to the set ($q_0 \neq 8$). These values of the scaling parameters can be used to analyse the Poincaré recurrences distribution function in kinetics which will be discussed in Chapter 16. In particular, one can expect that there can be more than one exponent that characterizes different distributions.

There are different observations that the recurrences distribution can be of the algebraic type (13.54) with $\gamma_{\text{rec}} \geq 3$ or $\alpha \geq 2$ if we apply the notations of the Feller's theorems with a corresponding Lévy index (*Note 15.6*).

One can expect more different deviations from (15.45) if we recall that the condition of independency of the recurrence times $\{t_j\}$ in Section 15.2 is a kind of approximation in dynamical systems with chaos. Typically, there are correlations between the neighbouring steps of Poincaré maps that are important for kinetics. Subsequently, there is a possibility of deviations from the Lévy distribution and Lévy process, and a more general approach is necessary. In other words, Lévy's idea of distributions with infinite moments can be valid for a fairly broad set of cases, but it does not mean that the infiniteness of the moments of distribution functions imposes the Lévy process. This also means there is a necessity to define a *flight* in some general way as to be able to provide a practical use of the notion in physical and mathematical senses.

In a very qualitative way, we can think about a phase space domain of motion Γ as strongly nonuniform area with small subdomains $\delta\Gamma_k$,

$$\sum_k \delta\Gamma_k \ll \Gamma, \tag{15.51}$$

such that the finite time positive Lyapunov exponents

$$\max \sigma_k \ll \sigma, \tag{15.52}$$

where σ is the Lyapunov exponent of the domain $\Gamma \setminus \sum_k \delta\Gamma_k$. Correspondingly, a piece of trajectory that passes a domain $\delta\Gamma_k$ will be called a *flight*. There are different flights of the same trajectory due to it passing through different subdomains of vanishing Lyapuniov exponent $\delta\Gamma_k$ and through the same subdomain

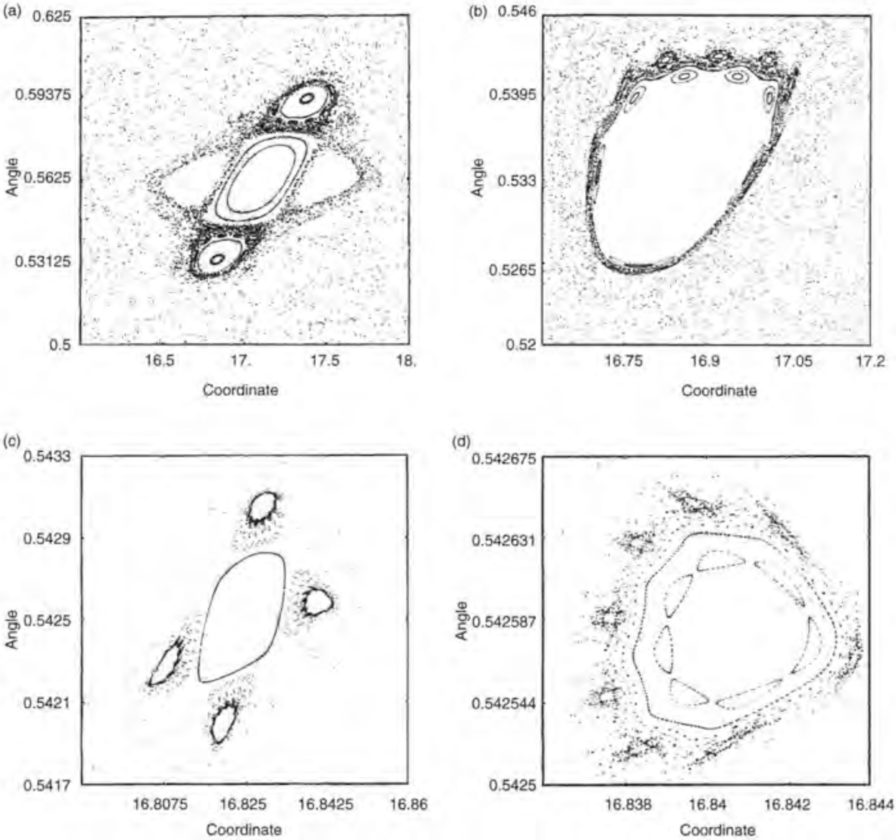

FIG. 15.4. An island and its vicinity for the Cassini billiard with $a = 4.030952$ and $c = 3$: (a) Poincaré plot of the initial island taken from Fig. 15.3(a); (b) magnification of the bottom island from (a); (c) magnification of the top left island from (b); and (d) magnification of the right island from (c).

at different time instants. A more accurate definition will be given in Chapter 21 through the notion of ϵ-separation of trajectories.

15.5 Weirstrass random walks (WRW)

The Weirstrass random walks (WRW) is perhaps the most appropriate model to understand simultaneously the origin of flights, their relation to the renormalization group equation, and the connection of the dynamics with HIT to the random processes with Bernoulli scaling. Having been coined by Shlesinger *et al.*, the WRW prepares a specific basis to understand the origin of fractional kinetics, fractal time, etc. (*Note 15.7*).

Consider a random walk on a one-dimensional periodic 'lattice' with a spacing equal to one and a probability p_j to make a step of the length a_j. Then probability density to make a step of the length ℓ is

$$P(\ell) = \frac{1}{2}\sum_{j=1}^{\infty} p_j[\delta(\ell - a_j) + \delta(\ell + a_j)], \tag{15.53}$$

if the random walk is symmetric, and

$$\int_{-\infty}^{\infty} d\ell\; P(\ell) = 1. \tag{15.54}$$

The crucial idea of the WRW is to consider only scaling type steps and corresponding probabilities, that is,

$$a_j = a^j, \quad p_j = Cp^j \tag{15.55}$$

with the normalization constant C

$$C = 1 - p. \tag{15.56}$$

Now $P(\ell)$ appears in the form

$$P(\ell) = \frac{1}{2}(1-p)\sum_{j=0}^{\infty} p^j[\delta(\ell - a^j) + \delta(\ell + a^j)]. \tag{15.57}$$

Using (15.57), we have for the second moment

$$\langle \ell^2 \rangle = \int_{-\infty}^{\infty} \ell^2 P(\ell) d\ell = (1-p)\sum_{j=0}^{\infty}(a^2 p)^j, \tag{15.58}$$

which diverges if $a^2 p \geq 1$. The characteristic function of $P(\ell)$ is

$$P(k) = \int_{-\infty}^{\infty} d\ell\; e^{ik\ell} P(\ell) = (1-p)\sum_{j=0}^{\infty} p^j \cos(ka^j), \tag{15.59}$$

which is the Weierstrass function and it explains the origin of the name WRW.

The function $P(k)$ satisfies the evident functional equation

$$P(k) = pP(ka) + (1-p)\cos k. \tag{15.60}$$

The solution of (15.60) can be presented as a sum

$$P(k) = P_s(k) + P_r(k) \tag{15.61}$$

of regular (holomorphic) $P_r(k)$ and singular $P_s(k)$ parts. The singular part $P_s(k)$ satisfies the renormalization equation

$$P_s(k) = p\; P_s(ak) \tag{15.62}$$

and its solution has a singular behaviour at $k = 0$.

The equation (15.60) is similar to the *renormalization group equation* (RGE) of the phase transition theory

$$F(g) = \ell^{-d} F(g') + G(g) \tag{15.63}$$

for free energy F, interaction constant g, renormalized interaction constant $g' = g'(g)$, regular part G of the free energy, renormalization length ℓ, and the system's dimension d. Nevertheless, (15.60) is simpler since its original explicit form (15.59) permits us to find $P(k)$ explicitly.

A qualitative analysis of the expression for $P(k)$ is based on the assumption that the form of the singular part $P_s(k)$ has the following behaviour near $k = 0$

$$P_s(k) = |k|^{\mu} Q(k) \tag{15.64}$$

with some exponent μ and nonsingular function $Q(k)$. That is sufficient to conclude that $0 < \mu < 2$ since the regular part of (15.59) has only even powers of k, and that $Q(k)$ should be periodic in $\ln k$ with a period $\ln a$ that follows from (15.62). After substitution of (15.64) in (15.62) we also obtain

$$\mu = \frac{|\ln p|}{\ln a}. \tag{15.65}$$

Let us recall that the Mellin transform $f_M(k)$ of a function $f(x)$ is defined as

$$f_M(s) = \int_0^{\infty} dk f(k) k^{s-1} \tag{15.66}$$

and the corresponding inverse equation is

$$f(k) = \int_{c-i\infty}^{c+i\infty} ds\ f_M(s) k^{-s} \tag{15.67}$$

(see Oberhettinger (1974)). Application of the Mellin transform to $f(k) = \cos(a^n k)$ gives

$$f_M(s) = \int_0^{\infty} dk \cos(a^n k) k^{s-1} = a^{-ns} \int_0^{\infty} d\xi \cos\xi \cdot \xi^{s-1}. \tag{15.68}$$

Substitution of (15.68) into (15.67) and (15.60) gives

$$\begin{aligned} P(k) &= \frac{1-p}{2\pi i} \sum_{j=0}^{\infty} p^j \int_{c-i\infty}^{c+i\infty} ds \frac{\Gamma(s)\cos(\pi s/2)}{a^{sj}|k|^s} \\ &= \frac{1-p}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \frac{\Gamma(s)|k|^{-s}\cos(\pi s/2)}{1-p/a^s} \end{aligned} \tag{15.69}$$

(see *Note 15.7* for references). The numerator has poles at $s = 0, -2, \ldots$, and the denominator has poles at

$$s_j = \frac{\ln p}{\ln a} \pm \frac{2\pi i j}{\ln a} = -\mu \pm \frac{2\pi i j}{\ln a}. \tag{15.70}$$

The final result from (12.69) and (12.70) is obtained by application of the residues over all poles:

$$P(k) = 1 + |k|^{\mu} Q(k) + p \sum_{n=1}^{\infty} \frac{(-1)^n k^{2n}}{(2n)!(1 - pa^{2n})},$$
$$Q(k) = \frac{p}{\ln a} \sum_{n=-\infty}^{\infty} \Gamma(s_n) \cos\left(\frac{\pi s_n}{2}\right) \exp\left(\frac{-2\pi i n \ln |k|}{\ln a}\right)$$

in full correspondence with (15.64) and (15.61).

The concluding remarks from the obtained result are fairly spectacular:

(i) The random walk model corresponds to the Bernoulli scaling and, at the same time, the model shows singularity (15.64) and infinite second moment;
(ii) The equation for the characteristic function (15.60) is similar to the renormalization group equation of the phase transition theory;
(iii) The singular part of the distribution function $Q(k)$ in (15.71) possesses slow oscillation with respect to $\ln k$, but not k. This is the so-called *log-periodicity* which will be one of the most important signatures of kinetics in chaotic systems (*Note 15.8*).

Notes

Note 15.1

The deviations from the 'Gaussianity' were mainly observed in numerical simulations although, from the very beginning, they were attributed to the presence of islands and cantori (Karney (1983); Chirikov and Shepelyansky (1984); Beloshapkin and Zaslavsky (1983); Geisel (1984); Ichikawa *et al.* (1987); Aizawa *et al.* (1989); Horita *et al.* (1990); and Zaslavsky *et al.* (1991)).

Note 15.2

Important theorems on the Lévy and Lévy-type distributions can be found in Gnedenko and Kolmogorov (1949); Feller (1949); Zolotarev (1986, 1997); Uchaikin and Zolotarev (1999); and Uchaikin (2003). Contemporary applications of the Lévy processes were analysed in Montroll and Shlesinger (1984); Bouchaud and Georges (1990); Geisel *et al.* (1987a, 1987b); and Geisel (1995). For applications to the chaotic dynamics, see Afanasiev *et al.* (1991); Zaslavsky (1992, 1994a,b); and Shlesinger *et al.* (1993, 1995a).

Note 15.3

See more on Lévy distribution in Lévy (1937); Feller (1949, 1957); Uchaikin and Zolotarev (1999).

Note 15.4

Different generalizations of the Lévy processes can be found in Uchaikin and Zolotarev (1999) and Uchaikin (2000). See also some example in Yanovsky *et al.* (2000).

Note 15.5

The results of simulation in Figs. 15.3 and 15.4 and in Table 15.1 are from Zaslavsky and Edelman (1997).

Note 15.6

These observations were obtained by simulation (see, for example, in Zaslavsky and Niyazov (1997); Zaslavsky *et al.* (1997); Rakhlin (2000); Leoncini and Zaslavsky (2002)). Another example, the Sinai billiard (Sinai (1963)) with infinite horizon, seems to have $\gamma = 3$ and $\alpha > 2$ up to a logarithmic factor that follows from a qualitative analysis and simulations (Bunimovich and Sinai (1973); Geisel *et al.* (1987b); Zacherl *et al.* (1986); Machta (1983); Machta and Zwanzig (1983); Zaslavsky and Edelman (1997)).

Note 15.7

The first paper on the WRW appeared in Shlesinger *et al.* (1981). This section follows the original work and the following publications in Hughes *et al.* (1981); Hughes *et al.* (1982); Shlesinger (1988); and Montroll and Shlesinger (1984). The connection to the RGE for the free energy was made in Shlesinger and Hughes (1981). The paper Hughes *et al.* (1982) showed that in higher dimensions the WRW leads to a lacunary series of spherical Bessel function for the generating function, which are cosines in the one-dimensional case. For the discussions of the WRW and dynamics, see Zaslavsky (2002b).

Note 15.8

The *log-periodicity* of the singular part $P_s(k)$ through the function $Q(k)$ with respect to k was well known in the RG theory of phase transitions (Niemeijer and van Leeuwen (1976)) but it did not play such an essential role as it did in the dynamical systems theory (Benkadda *et al.* (1999); Zaslavsky (2000a, 2002b)). For other applications of the log-periodicity see Sornette (1998)).

Problems

More complicated problems are marked by (*).

15.1 Prove (15.23) using the chain property (15.21).

*15.2** Prove asymptotics (15.34) using the steepest descent method.

15.3 Derive the equation (15.69) replacing summation and integration.